Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

APPLICATION OF GROUPS TO A COMPLEX PROBLEM IN ARRANGEMENTS

BY L. E. DICKSON

1. Viewed as a problem in configurations, the general problem here considered possesses especial interest. The application of group-theory to problems of this type affords not only a better insight into the essential features of the problem but also a method for the rapid construction of solutions and, what is even more essential in the longer solutions, affords simultaneously a proof of their correctness.

In its initial form* the problem† related to the seating of m persons at a round table on successive days such that no one shall sit twice between the same two companions. The precise formulation of the general problem will be preceded by some definitions and explanations and the complete consideration of the example $m = 5$.

While we are concerned with a ring and not a row, nevertheless we shall, for convenience in printing, write the symbols in a straight row and not in ring form. Then in $ABCDE$ it is understood that E is followed by A. It being a question of relative order only, we regard as *equivalent* a ring and the one derived from it by reversing the order of its symbols. Then, for $m = 5$, the following ten arrangements are equivalent:

$$\begin{array}{lllll} ABCDE, & BCDEA, & CDEAB, & DEABC, & EABCD \\ AEDCB, & BAEDC, & CBAED, & DCBAE, & EDCBA. \end{array}$$

In general, the $2m$ arrangements derived from a given one by applying the substitutions of the dihedron group G_{2m} are regarded as *equivalent*. The $2m$

---

* Proposed in the *American Mathematical Monthly*, 1899, p. 92, by Professor C. H. Judson, who gave examples for $m = 6$ and $m = 8$ in the *Monthly*, 1900, p. 72. Another example for $m = 6$ and one for $m = 7$ by Dr. F. H. Safford will appear soon in the *Monthly*. The two examples for $m = 6$, whose correctness I have verified, are here reproduced.

† I give an equivalent problem, to emphasize the notion of dihedrons: In each of a set of circular discs are bored m holes of different shape at the vertices of a concentric regular polygon (the latter the same for each disc). It is required to determine all sequences of the holes on the various discs such that it shall be impossible to place two discs in a position where three consecutive holes of the same shape are juxtaposed.

arrangements will be said to define the same *dihedron*,* which may be represented by any one of the $2m$ arrangements. However, we shall employ always one of the two arrangements which begins with a particular letter, as A.

Two dihedrons, as $ABCDE$ and $ABECD$, will be called *consistent* when there is no letter which occurs in both between the same two letters.

For $m=5$, the only dihedrons consistent with $ABCDE$ are

$$ABECD, \quad ABDEC, \quad ACBED, \quad ACDBE, \quad ADBCE \quad ACEBD.$$

Any two of the first five are consistent, while no one of them is consistent with the sixth.

A set of k mutually consistent dihedrons on m letters will be called a *complete set*, S_k^m, when every further dihedron on the m letters is inconsistent with at least one dihedron of S_k^m. Hence for $m = 5$ the only complete sets are

$$S_6^5 = \{ABCDE, ABDEC, ABECD, ACBED, ACDBE, ADBCE\},$$

$$S_2^5 = \{ABCDE, \quad ACEBD\}.$$

The general problem is to find all complete sets S_k^m of dihedrons. The maximum value of k is $\frac{1}{2}(m-1)(m-2)$, this being the number of pairs of letters between which a given one may lie. For this maximum, k, a set S_k^m possesses remarkable symmetry. It may be given a form, using the symbols $A_1, \cdots, A_m$, such that there are exactly $m - i$ dihedrons $A_1 A_2 \cdots$, for $i = 2, \cdots, m-1$. This is exemplified in S_6^5 above and in the following complete sets S_{10}^6:

(Judson)	(Safford)	(Dickson)
$ABCDEF$	$ABCDEF$	$ABCDEF$
$ABDCFE$	$ABDCFE$	$ABDFEC$
$ABEFDC$	$ABEDFC$	$ABECFD$
$ABFECD$	$ABFECD$	$ABFDCE$
$ACFDBE$	$ACDFBE$	$ACDBFE$
$ACEBFD$	$ACEBDF$	$ACFEBD$
$ACBEDF$	$ACBEFD$	$ACBEDF$
$ADECBF$	$ADECBF$	$ADECBF$
$ADBFCE$	$ADBFCE$	$ADCFBE$
$AEDBCF$	$AEDBCF$	$AEDBCF$

---

* This corresponds to the dihedron in geometry, that is, a regular polygon whose plane has two aspects, with order attached to the vertices. If, on viewing the polygon from a point above its plane, the vertices are read in a certain order, then from a point below the plane they will appear in the reverse order.

In view of the particular interest of the sets with maximum k, we will be concerned here* with *the special problem to find all maximum complete sets* S_k^m, *where* $k = \frac{1}{2}(m-1)\ (m-2)$.

2. Before undertaking in general the question of the applicability of group theory to the present problems, we will consider two of the simplest cases.

For $m=4$ the distinct dihedrons are $ABCD$, $ABDC$, $ACBD$, and these are consistent. They are equivalent to $ADCB$, $ACDB$, $ADBC$, respectively, each having the symbols of the corresponding one in reverse order. The six give all permutations of B, C, D, and hence define the symmetric group on three letters.

In S_6^5 the dihedrons are obtained from $ABCDE$ by the substitutions:

$$I = \text{identity},\ (CDE),\ (CED),\ (BC)(ED),\ (BCD),\ (BDC).$$

These do not form a group since $(CDE)(BDC) = (BDE)$. Taking, however, the second notation $AEDCB$, $\cdots$, for each of the 6 dihedrons, we find that they are obtained from $ABCDE$ by the substitutions

$$(BE)(CD),\ (BCE),\ (BDE),\ (BD)(EC),\ (BEC),\ (BED).$$

These together with the preceding six substitutions form the alternating group on 4 letters. Extending it by $(ABCDE)$, to pass from rings to rows, we obtain† the alternating groups on 5 letters.

---

* I have made considerable headway on the *general* problem for $m = 6$. The method begins as for $m = 5$, there being for $m = 6$ exactly 38 dihedrons consistent with $ABCDEF$. The number of cases is greatly reduced by transformation by suitable substitutions. The aim is to obtain a final list of S_k^6 in which no two differ by a mere change in notation, *i. e.* are not conjugate under the symmetric group on 6 letters. I give a few non-maximum complete sets:

$$S_8^6 = \{ABCDEF,\ ABDFEC,\ ABEFCD,\ ABFDCE,\ ACDBFE,\ ACFBED,\ ADBECF,\ ADFCBE\};$$

$$S_7^6 = \{ABCDEF,\ ABDFEC,\ ABECFD,\ ABFDCE,\ ACDBFE,\ ACFBED,\ AEDBCF\};$$

$$S_7^6 = \{ABCDEF,\ ABDCFE,\ ABECFD,\ ABFDEC,\ AECBDF,\ AFCBED,\ AEDBFC\};$$

$$S_7^6 = \{ABCDEF,\ ABDFEC,\ ABEFCD,\ ABFDCE,\ ACBEDF,\ ADBECF,\ ACDBFE\},$$

where for each S_7^6 there is a single complete set having the first six dihedrons.

† This might have been anticipated since of the two substitutions

$$\begin{pmatrix} ABCDE \\ \alpha\,\beta\,\gamma\,\delta\,\epsilon \end{pmatrix} = S,\ \begin{pmatrix} ABCDE \\ \alpha\,\beta\,\gamma\,\epsilon\,\delta \end{pmatrix} = S(\delta\epsilon),$$

one is even and one odd.

3. For the general case of $S^m_{\frac{1}{2}(m-1)(m-2)}$, we likewise take each dihedron in direct and reverse order, so that we have $(m-1)(m-2)$ arrangements (rings) beginning with a fixed letter A. The group $G_{(m-1)(m-2)}$ of the resulting substitutions must be doubly-transitive since we have the arrangements $AX \cdots Y$, where X and Y are arbitrary. Hence[*] $m-1$ is the power of a prime p^n. Doubly transitive groups of degree p^n and order $p^n(p^n-1)$ have been completely enumerated[†] for general p and $n=1, 2, 3$, and[‡] for $p^n=2^4$, the result presumably[§] holding also for $n > 3$. There is a single type of doubly-transitive group $G^{p^n}_{p^n(p^n-1)}$, except for $p^n = 3^2$ when there is a second type H^9_{72}. The first may be taken concretely as the group of all linear transformations

$$(1) \qquad S_{a,b}: \qquad z' = az + b \qquad (a \neq 0),$$

with coefficients in the Galois Field of order p^n. Allowing z to run through the series of marks of this $GF[p^n]$, we obtain a group on p^n letters, which is doubly-transitive since we may solve

$$z_1' = az_1 + b, \qquad z_2' = az_2 + b \qquad (z_1 \neq z_2,\ z_1' \neq z_2')$$

for a and b in the field, $a \neq 0$. The additional type is[¶]

$$(2) \qquad H^9_{72} = \{ (abcd)(efgh), \quad (ahcf)(bgde), \quad (iac)(bgh)(dfe) \}$$

Considering the m arrangements in each ring, we extend the preceding groups by the cyclic substitution $(A \cdots)$, and obtain triply-transitive groups. From G, we obtain the group of the $(p^n+1)\,p^n\,(p^n-1)$ linear fractional transformations

$$(3) \qquad z' = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0).$$

From H^9_{72} we obtain Cole's group 720_3 of order 720 on 10 letters.[**] The preceding types of doubly and triply transitive groups are all of the specified orders that occur in the lists of substitution groups on m letters, $m \gtreqless 17$.

---

[*] Jordan, Récherches sur les substitutions, *Journal de Mathématiques*, ser. 2, vol. 17, (1872), pp. 351–367. (*Cf.* next reference).

[†] Burnside, On doubly-transitive groups of degree n and order $n(n-1)$, *Messenger of Mathematics*, vol. 25 (1895–96), pp. 147–153.

[‡] Miller, *American Journal*, vol. 20 (1898), p. 231.

[§] If further exceptions occur, they arise for $p^n \geqq 81$, so that their use in the present problem would be impracticable.

[¶] Cole's (iii), *Bulletin*, 1893, p. 257. His (ii) is G^9_{72}.

[**] *Quarterly Journal*, vol. 27 (1895), p. 44. His 720_2 is the group of (3).

4. *Application of the linear group.* Since the linear fractional group is triply-transitive, we may assume that the desired complete set, S_k^m, where $m = p^n + 1$ and $k = \frac{1}{2}p^n(p^n - 1)$, contains the dihedron*

$$(4) \qquad \infty \quad 0 \quad 1 \quad a_3 \quad a_4 \quad \cdots \quad a_{p^n},$$

where $a_1 = 0$, $a_2 = 1$, $a_3, a_4, \cdots, a_{p^n}$ give the p^n distinct marks of the $GF[p^n]$. Now $S_{r,s}$, defined by (1), replaces dihedron (4) by

$$(5) \qquad \infty \quad s \quad r+s \quad a_3 r + s \quad a_4 r + s \quad \cdots \quad a_{p^n} r + s.$$

We impose first the condition that there shall exist a linear transformation T, of the form (1), which replaces (4) by

$$(6) \qquad \infty \quad a_{p^n} \quad \cdots \quad a_4 \quad a_3 \quad 1 \quad 0,$$

the latter being a second notation for the dihedron (4). Hence, if $p > 2$,

$$(7) \quad T = (0\, a_{p^n})(1\, a_{p^n-1})(a_3 a_{p^n-2}) \cdots (a_i a_{p^n-i+1}) \cdots (a_{\frac{1}{2}(p^n-1)}\, a_{\frac{1}{2}(p^n+3)})$$

leaving fixed $a_{\frac{1}{2}(p^n+1)}$. If $p = 2$, the final transposition in T is $(a_{2^{n-1}}\, a_{2^{n-1}+1})$, and no letter remains fixed. Thus T is of period 2 and hence of the form

$$(8) \qquad T: \quad z' = -z + a_{p^n}.$$

The conditions that (8) shall represent (7) are, since $a_2 = 1$,

$$(9) \qquad a_{p^n-i} = a_{p^n} - a_{i+1}, \qquad [i = 1, \cdots, q],$$

where $q = \frac{1}{2}(p^n - 1)$ if $p > 2$, $q = 2^{n-1} - 1$ if $p = 2$.

We impose next the condition that the dihedrons (5), with r, s, ranging over the $GF[p^n]$, $r \neq 0$, shall form a complete set. Taking two of these dihedrons, arising from $S_{r,s}$ and $S_{\rho,\sigma}$, we look for both forward and backward coincidences in triples. But if a linear transformation S replaces $\infty\, a_1 \cdots a_{p^n}$ by $\infty\, \beta_1 \cdots \beta_{p^n}$, and if $\{a_2, a_3, a_4\} \equiv \{\beta_3, \beta_2, \beta_1\}$, for example, then the product TS replaces $\infty\, a_1 \cdots a_{p^n}$ by $\infty\, \beta_{p^n} \cdots \beta_1$ so that for the latter $\{a_2, a_3, a_4\} \equiv \{\beta_1, \beta_2, \beta_3\}$. Hence, unless ∞ enters, we need only look for forward coincidences in triples. When ∞ enters, the only possible backward coincidence is

---

* The element ∞ here adjoined to the field is used in the sense of *Linear Groups*, p. 260.

$$\{a_{p^n}\, r + s,\ \infty,\ s\} \equiv \{\sigma,\ \infty,\ a_{p^n}\, \rho + \sigma\},$$

requiring $\rho = -r$, $\sigma = a_{p^n}\, r + s$, whence

$$S_{\rho,\sigma} = TS_{r,s}.$$

But we retain only one of the arrangements resulting from the two transformations $S_{r,s}$ and $TS_{r,s}$. Hence there here results no condition on the a_i.

We proceed to examine the forward coincidences in triples. If

$$\{\rho,\ \rho + \sigma,\ a_3\rho + \sigma\} \equiv \{r + s,\ a_3 r + s,\ a_4 r + s\},$$

then $\rho = (a_3 - 1)r$, $a_3\rho = (a_4 - 1)r$. Then if

$$a_3(a_3 - 1) \neq a_4 - 1, \tag{10}$$

we would have $r = \rho = 0$, $s = \sigma$, and the two dihedrons in question would be identical. Hence (10) is a sufficient condition that the specified coincidence in triples does not arise for distinct dihedrons. It is also a necessary condition. For if (10) does not hold, we may determine ρ and σ in terms of r and s, so that the coincidence shall arise. Likewise, to avoid a coincidence

$$\{\sigma,\ \rho + \sigma,\ a_3\rho + \sigma\} \equiv \{a_i r + s,\ a_{i+1} r + s,\ a_{i+2} r + s\}, \qquad (3 \leqq i \leqq p^n \quad 2)$$

we have the necessary and sufficient conditions

$$a_3(a_{i+1} - a_i) \neq a_{i+2} - a_i \qquad (i = 3, \cdots, p^n - 2). \tag{11}$$

To avoid the coincidence of $\{\rho + \sigma,\ a_3\rho + \sigma,\ a_4\rho + \sigma\}$ with

$$\{s,\ r + s,\ a_3 r + s\}, \quad \{a_i r + s,\ a_{i+1} r + s,\ a_{i+2} r + s\},$$

we obtain, respectively, conditions (10) and

$$\begin{vmatrix} a_3 - 1 & a_{i+1} - a_i \\ a_4 - 1 & a_{i+2} - a_i \end{vmatrix} \neq 0 \qquad (i = 3, \cdots, p^n - 2). \tag{12}$$

To avoid the coincidence of $\{a_i\rho + \sigma,\ a_{i+1}\rho + \sigma,\ a_{i+2}\rho + \sigma\}$ with

$$\{s,\ r + s,\ a_3 r + s\},\ \{r + s,\ a_3 r + s,\ a_4 r + s\},\ \{a_j r + s,\ a_{j+1} r + s,\ a_{j+2} r + s\},$$

we obtain, respectively, conditions (11), (12), and

$$\begin{vmatrix} a_{i+1} - a_i & a_{j+1} - a_j \\ a_{i+2} - a_i & a_{j+2} - a_j \end{vmatrix} \neq 0 \qquad (i, j = 3, \cdots, p^n - 2;\ i < j). \tag{13}$$

Finally, since 0, 1, $a_3, \cdots, a_{p^n}$ are the distinct marks of the field,

$$\}\infty, \sigma, \rho + \sigma\{ \neq \}\infty, s, r + s\{, \ \}a_{p^n}\rho + \sigma, \infty, \sigma\{ \neq \}a_{p^n} r + s, \infty, s\{,$$
$$\}a_{p^n - 1}\rho + \sigma, \ a_{p^n}\rho + \sigma, \infty\{ \neq \}a_{p^n - 1} r + s, \ a_{p^n} r + s, \infty\{.$$

We have now established the following

THEOREM. *The distinct* dihedrons* (5) *derived from the dihedron* (4) *upon applying the transformations* (1) *of the linear group in the* $GF[p^n]$ *form a complete set* S^m_k, *where* $m = p^n + 1$ *and* $k = \frac{1}{2}p^n(p^n - 1)$, *if and only if conditions* (9), (10), (11), (12), *and* (13) *hold.*†

5. *The example* $p^n = 5$. The a_i are now integers taken modulo 5. Conditions (9) now give $a_4 \equiv a_5 - 1$, $a_3 \equiv 3a_5 \pmod 5$. Then (10) becomes

$$3a_5(3a_5 - 1) \neq 2(3a_5 - 1), \ i.\,e., \ a_5 \neq 2 \text{ or } 4 \pmod 5.$$

But a_3, a_4, a_5 form a permutation of 2, 3, 4. Hence $a_5 \equiv 3$. Thus $a_3 \equiv 4$, $a_4 \equiv 2$. There is now a single condition (11) and a single (12), each given by $i = 3$; but no (13). The first two reduce to $-3 \neq -1$, $-1 \neq 0$. Hence for $p^n = 5$, group theory furnishes one and but one complete set‡ S^6_{10}:

$S_{1,0}$:	∞	0	1	4	2	3	$S_{1,1}$:	∞	1	2	0	3	4
$S_{4,0}$:	∞	0	4	1	3	2	$S_{4,1}$:	∞	1	0	2	4	3
$S_{2,0}$:	∞	0	2	3	4	1	$S_{3,4}$:	∞	4	2	1	0	3
$S_{3,0}$:	∞	0	3	2	1	4	$S_{1,4}$:	∞	4	0	3	1	2
$S_{2,1}$:	∞	1	3	4	0	2	$S_{2,4}$:	∞	2	4	0	1	3

It is immediately identified with Judson's solution (§ 1).

---

* Employing only one of each pair resulting from $S_{r,s}$, $TS_{r,s}$.

† We do not discuss in general the question of the existence of at least one set of distinct marks a_i satisfying the specified conditions. As shown in the examples, such sets exist for $p^n = 5, 7, 9, 11$, and their number increases rapidly with p^n. By §§ 1, 2, there is a single set for $p^n = 3$ or 4, as may now be verified directly. For example, for $p^n = 2^2$, the marks are 0, 1, j, $j + 1$, where $j^2 \equiv j + 1 \pmod 2$. Since $j + 1$ satisfies the same congruence, we may by choice of the notation, set $a_3 = j$, $a_4 = j + 1$. Then (10) and the single condition (9) are satisfied, while (11), (12), (13) do not occur.

‡ It is transformed into a complete set with the first dihedron ∞ 0 1 2 3 4 by (2 3 4). The analytic representation of (2 3 4) is

$$z' \equiv 3z^3 - z^2 - z \pmod 5.$$

The group of the resulting complete set is therefore not linear.

6. *The example* $p^n = 7$. Now a_3, a_4, a_5 a_6, a_7 form a permutation of 2, 3, 4, 5, 6. Conditions (9) give

$$(9') \qquad a_6 \equiv a_7 - 1, \quad a_5 \equiv a_7 - a_3, \quad a_4 \equiv 4a_7 \pmod 7.$$

Conditions (11) for $i = 3, 4, 5$ become (modulo 7)

$$\begin{array}{lll} a_3(4a_7 - a_3) \neq a_7 - 2a_3, & \text{or} & (4a_3 - 1)(a_7 - 2a_3) \neq 0, \\ a_3(4a_7 - a_3) \neq 4a_7 - 1, & \text{or} & (a_3 - 1)(4a_7 - a_3 - 1) \neq 0, \\ a_3(a_3 - 1) \neq a_3, & \text{or} & a_3(a_3 - 2) \neq 0. \end{array}$$

Hence $a_3 \neq 2$, $a_7 \neq 2a_3$, $a_7 \neq 2\,a_3 + 2$. The three conditions (12) become

$$(4a_7 - a_3)(a_3 - 2a_7 + 3) \neq 0, \quad (4a_7 - 1)(a_3 - 2a_7 + 3) \neq 0,$$
$$(a_3 - 1)(a_3 - 4a_7 + 1) \neq 0,$$

whence $a_7 \neq 4a_3 + 5$, $a_7 \neq 2$. The three conditions (13) become

$$(4a_7 - a_3)(3a_7 + 2a_3 - 1) \neq 0, \quad (4a_7 - a_3)(3a_3 + 1) \neq 0,$$
$$4a_7 - a_3^2 + a_3 - 1 \neq 0,$$

the last being (10). Hence the only conditions on the a_i are (9′), (10), and

$$a_3 \neq 2, \quad a_7 \neq 2, \quad a_7 \neq 2a_3 \quad a_7 \neq 2a_3 + 2, \quad a_7 \neq 4a_3 + 5.$$

If $a_3 = 3$, then a_7 differs from 0, 1, 2, 3, 6, while $a_7 = 4$ gives $a_6 = 3 = a_3$ and is excluded. Proceeding similarly for $a_3 = 4$, 5, 6, we obtain only the following admissible systems a_i:

a_3	a_4	a_5	a_6	a_7
3	6	2	4	5
4	3	2	5	6
6	5	4	2	3
6	2	5	3	4

The second system leads to the complete set* S_{21}^{8} :

---

* It is immediately identified with Judson's example for 8 persons.

	∞ 0 1 4 3 2 5 6		
$S_{4,0}$:	∞ 0 4 2 5 1 6 3	$S_{2,4}$:	∞ 4 6 5 3 1 0 2
$S_{3,0}$:	∞ 0 3 5 2 6 1 4	$S_{5,4}$:	∞ 4 2 3 5 0 1 6
$S_{2,0}$:	∞ 0 2 1 6 4 3 5	$S_{1,4}$:	∞ 4 5 1 0 6 2 3
$S_{5,0}$:	∞ 0 5 6 1 3 4 2	$S_{6,4}$:	∞ 4 3 0 1 2 6 5
$S_{6,0}$:	∞ 0 6 3 4 5 2 1	$S_{5,3}$:	∞ 3 1 2 4 6 0 5
$S_{4,1}$:	∞ 1 5 3 6 2 0 4	$S_{1,3}$:	∞ 3 4 0 6 5 1 2
$S_{3,1}$:	∞ 1 4 6 3 0 2 5	$S_{4,3}$:	∞ 3 0 5 1 4 2 6
$S_{2,1}$:	∞ 1 3 2 0 5 4 6	$S_{4,2}$:	∞ 2 6 4 0 3 1 5
$S_{5,1}$:	∞ 1 6 0 2 4 5 3	$S_{3,2}$:	∞ 2 5 0 4 1 3 6
$S_{6,1}$:	∞ 1 0 4 5 6 3 2	$S_{6,5}$:	∞ 5 4 1 2 3 0 6

The remaining three systems furnish complete sets different from the preceding. Whether they are themselves distinct is not discussed here.

7. *The example* $p^n = 3^2$. As the marks a_i of the $GF[3^2]$, we may take $0, \pm 1, \pm i, \pm i + 1, \pm i - 1$, where $i^2 \equiv -1 \pmod 3$. Then the only systems of solutions a_i of (9), (10), (11), (12), (13) are found to be the four:

a_3	a_4	a_5	a_6	a_7	a_8	a_9
$\pm i + 1$	$\mp i - 1$	$\pm i - 1$	-1	$\pm i$	$\mp i$	$\mp i + 1$
$\pm i + 1$	$\mp i - 1$	$\mp i$	$\mp i + 1$	-1	$\pm i - 1$	$\pm i$

8. *Application of the isolated group* H_{72}^9. This group (2) has a self-conjugate subgroup G of order 9 composed of the substitutions C, D, E, F (given below) and their powers. We may form a table of the distinct substitutions H_{72}^9 with those of G in the first row (here printed as first column), the left-hand multipliers being $I =$ identity, A, $A^2 = B^2$, A^3, B, B^3, AB, $BA = A^3B = (AB)^{-1}$.

(1)	(2)	(3)
Identity	$A = (abcd)(efgh)$	$A^2 = (ac)(bd)(eg)(fh)$
$C = (aci)(bgh)(dfe)$	$AC = (agbi)(cfhd)$	$A^2C = (ai)(bf)(dg)(eh)$
$C^2 = (aic)(bhg)(def)$	$AC^2 = (ahfb)(cedi)$	$A^2C^2 = (be)(ci)(dh)(fg)$
$D = (bdi)(agf)(che)$	$AD = (adge)(bhci)$	$A^2D = (ah)(bi)(cg)(ef)$
$D^2 = (bid)(afg)(ceh)$	$AD^2 = (aidf)(begc)$	$A^2D^2 = (ae)(cf)(di)(gh)$
$E = (ige)(cbf)(ahd)$	$AE = (afec)(dhig)$	$A^2E = (ab)(ch)(df)(gi)$
$E^2 = (ieg)(cfb)(adh)$	$AE^2 = (achg)(bfie)$	$A^2E^2 = (af)(bh)(cd)(ei)$
$F = (ifh)(cdg)(aeb)$	$AF = (bdeh)(cgif)$	$A^2F = (ad)(bg)(ce)(fi)$
$F^2 = (ihf)(cgd)(abe)$	$AF^2 = (aeih)(bgfd)$	$A^2F^2 = (ag)(bc)(de)(hi)$

(4)

$A^3 = (adcb)(ehgf)$

$A^3C = (afdi)(bcge)$

$A^3C^2 = (aegd)(bich)$

$A^3D = (aibg)(cdhf)$

$A^3D^2 = (abfh)(cide)$

$A^3E = (bhed)(cfig)$

$A^3E^2 = (ahie)(bdfg)$

$A^3F = (aghc)(beif)$

$A^3F^2 = (acef)(dgih)$

(5)

$B = (ahcf)(bgde)$

$BC = (abhi)(cegf)$

$BC^2 = (ageh)(cdfi)$

$BD = (aedc)(bfgi)$

$BD^2 = (acgb)(idhe)$

$BE = (adig)(befh)$

$BE^2 = (biec)(fdgh)$

$BF = (aife)(hdbc)$

$BF^2 = (afbd)(hgci)$

(6)

$B^3 = (afch)(bedg)$

$B^3C = (aefi)(bdhc)$

$B^3C^2 = (adbf)(cghi)$

$B^3D = (bcei)(dfhg)$

$B^3D^2 = (agid)(bhfe)$

$B^3E = (acde)(bigf)$

$B^3E^2 = (abgc)(dieh)$

$B^3F = (aheg)(cifd)$

$B^3F^2 = (aihb)(cfge)$

(7)

$AB = (agce)(bfdh)$

$ABC = (ahgi)(becd)$

$ABC^2 = (abdg)(cfei)$

$ABD = (afib)(degh)$

$ABD^2 = (bgef)(chid)$

$ABE = (aehf)(bcig)$

$ABE^2 = (aied)(cgfh)$

$ABF = (acbh)(difg)$

$ABF^2 = (adfc)(bihe)$

(8)

$A^3B = (aecg)(bhdf)$

$A^3BC = (adei)(chfg)$

$A^3BC^2 = (afhe)(bgic)$

$A^3BD = (acfd)(behi)$

$A^3BD^2 = (ahbc)(dgfi)$

$A^3BE = (aigh)(bdce)$

$A^3BE^2 = (agdb)(cief)$

$A^3BF = (abif)(dhge)$

$A^3BF^2 = (bfeg)(cdih)$

We seek the conditions under which these 72 substitutions will give rise to a complete set S_{36}^{10} of dihedrons each occurring under two notations (§1) as follows:

$$j\,a\,b\,\alpha\,\beta\,\gamma\,\delta\,\epsilon\,\rho\,\sigma, \qquad j\,\sigma\,\rho\,\epsilon\,\delta\,\gamma\,\beta\,\alpha\,b\,a, \tag{14}$$

where j is a new (tenth) letter. We may assume that a and b occupy the second and third places, since H_{72}^9 is doubly-transitive. The substitution which interchanges the two sets (14) is

$$T = (a\sigma)\ (b\rho)\ (\alpha\epsilon)\ (\beta\delta)\ (\gamma). \tag{15}$$

It must belong to H_{72}^9 by hypothesis. It cannot be A^2E, which interchanges a and b; nor A^2D^2 which leaves b fixed; nor A^2C^2 which leaves a fixed. Hence T must be one of the remaining six of period 2:

$$A^2,\ A^2C,\ A^2D,\ A^2E^2,\ A^2F,\ A^2F^2. \tag{16}$$

We consider here* only the case $T = A^2$. Then

$$\sigma = c,\ \rho = d,\ \gamma = i,\ (\alpha\epsilon)\ (\beta\sigma) = (eg)\ (fh).$$

Now $\alpha \neq e$, since A^2D^2 replaces the triple *abe* by *eba*. Also $\epsilon \neq e$, when $\delta = f$, since A^2E replaces *fedc* by *defh*. We consider in detail the case $\alpha = g$, $\delta = h$. Then the first dihedron (14) is *jabgfihedc*. Applying to it the 36 substitutions† in the blocks (1), (2), (5), (7) of the above table, we obtain the following 36 dihedrons, written in the same order:

jabgfihedc
jcgheabdfi
jihbdcgfea
jgdfabecih
jfiagdchbe
jhfecgdiab
jdcibeaghf
jeachfibgd
jbedihfacg
jbchgiefad
jgibhadecf
jhagbcfdie
jdhefbcagi
jiecadhgfb
jfbdegicha
jcfaiegbdh
jadicfbheg
jegfdhaibc
jhgdaicbef
jbhfcaigde
jgbeicahfd
jefigbhdca
jcabfdeihg
jdeahgbfic
jaihdefcgb
jicgefdabh
jfdcbhgeai
jgfcdibahe
jheifagcbd
jbdaechigf
jfahibdgec
jagebdifch
jecbagfhdi
jibfhecdag
jchdgfaeib
jdigchebfa

This set forms a complete set S_{36}^{10}, as a tedious verification‡ shows.

THE UNIVERSITY OF CHICAGO,
FEBRUARY, 1904.

---

* An exhaustive treatment should make use of normalizations by transformations within H_{72}^9 so that only independent complete sets shall result. For instance, the 9 substitutions of period 2 are all conjugate within H.

† These together with their products on the left by $T = A$ give the 72 distinct substitutions of H_{72}^9, so that they give all the distinct dihedrons arising.

‡ When so many dihedrons enter as here, the verification may best be made by listing all the forward and backward triples entering. Thus

$$aj \} b,\ c,\ d,\ e,\ f,\ g,\ h,\ i \{ \qquad ab \} c,\ d,\ e,\ f,\ g,\ h,\ i,\ j \{$$

are two of the 81 lines in the list for the present complete set.

ADDENDUM. (AUGUST, 1904).

9. From the conditions (§4) for a set of marks a_i we may derive a more elegant set of equivalent conditions, by means of which the numerical computations for a complete solution are greatly reduced.

Conditions (10), (11), (12), (13) may be combined into the set

$$\begin{vmatrix} a_{i+1} - a_i & a_{j+1} - a_j \\ a_{i+2} - a_i & a_{j+2} - a_j \end{vmatrix} \neq 0 \qquad (i, j = 1, 2, \ldots, p^n - 2 ;\ i < j),$$

since $a_1 = 0$, $a_2 = 1$. Hence they are the conditions that the functions

$$f_i \equiv (a_{i+2} - a_i)/(a_{i+1} - a_i) \qquad (i = 1, 2, \cdots, p^n - 2)$$

shall be distinct. Since the a's are distinct, $f_i \neq 0, 1$. Hence conditions (10)–(13) require merely that $f_1, \cdots, f_{p^n-2}$ shall form a permutation of the marks $\neq 0, 1$. Set

$$g_i = f_i - 1, \qquad \pi_i = g_1 g_2 \cdots g_{i-1} g_i.$$

By induction it is readily shown that

$$(a) \qquad a_t = a_{t-1} + \pi_{t-2} \qquad (t = 3, 4, \cdots, p^n).$$

Indeed, the formula is true for $t = 3$ since $a_3 = a_2 + g_1 = f_1$. Next,

$$a_{t+1} = a_{t-1} + f_{t-1}\ (a_t - a_{t-1}) = a_t - \pi_{t-2} + f_{t-1}\, \pi_{t-2} =$$
$$a_t + \pi_{t-2}\, g_{t-1} = a_t + \pi_{t-1},$$

so that the formula is true for $t + 1$ if true for t. It follows that

$$a_{t+1} = a_{t-1} + \pi_{t-2} f_{t-1}.$$

From the latter it follows by induction from a_j to a_{j+2} that

$$(b) \qquad a_{2k} = 1 + \sum_{i=1}^{k-1} \pi_{2i-1} f_{2i}, \qquad a_{2k+1} = f_1 + \sum_{i=1}^{k-1} \pi_{2i} f_{2i+1}.$$

Finally, we obtain the conditions on the g_i resulting from (9). We note first that the set (9) is equivalent to the set

$$a_{p^n} - a_{p^n-1} = 1,\ a_{p^n-i+1} - a_{p_n-i} = a_{i+1} - a_i \qquad (i = 2, \cdots, q),$$

the first being (9) for $i = 1$, and the second being derived by subtracting (9) for $i = j$ from (9) for $i = j - 1$. In view of (a), these become

$$\pi_{p^n-2} = 1, \quad \pi_{p^n-i-1} = \pi_{i-1} \qquad (i = 2, \cdots, q).$$

Hence we get $$F_i \equiv g_i\, g_{i+1} \cdots g_{p^n-i-1} = 1 \qquad (i = 1, 2, \cdots, q).$$

Dividing F_i by F_{i+1}, we get

$$(c) \qquad g_i\, g_{p^n-i-1} = 1 \qquad (i = 1, 2, \cdots, q-1).$$

Also $F_q = 1$ becomes

$$(d) \qquad g_{\frac{1}{2}(p^n-1)} = 1 \quad \text{if } p > 2, \qquad g_{2^{n-1}-1}\, g_{2^{n-1}} = 1 \ \text{if } p = 2.$$

Evidently relations (c) and (d) are together equivalent to the set $F_i = 1$.

THEOREM. *Every set of distinct marks* $a_1 = 0$, $a_2 = 1$, a_i $(i = 3, \cdots, p^n)$ *which satisfy conditions* (9) – (13) *can be determined as follows: If* $p>2$, *we take in succession as the* g_i $[i = 1, \cdots, \frac{1}{2}(p^n - 3)]$ *each** *set of marks for which the* $p^n - 3$ *marks* g_i, $1/g_i$ $[i = 1, \cdots, \frac{1}{2}(p^n - 3)]$ *form a permutation of all the marks* $\neq 0, 1, -1$, *and take* $g_{\frac{1}{2}(p^n-1)} = 1$. *We then compute the* a_i *by* (b), *or more easily by* (a). *We retain only the sets in which the* a_i *are all distinct. If* $p = 2$, *we take as the* g_i $(i = 1, \cdots, 2^{n-1} - 1)$ *each set of marks for which the* $2^n - 2$ *marks* g_i, $1/g_i$ $(i = 1, \cdots, 2^{n-1} - 1$ *form a permutation of all the marks* $\neq 0, 1$.

For $p^n = 11$, the computation was performed in about five hours, including repeated checking. The work was done in tabular form with columns headed

$$g_1, \cdots, g_9, \ \pi_2 = \pi_7 = g_1 g_2, \ \pi_3 = \pi_6 = g_1 g_2 g_3, \ \pi_4 = \pi_5 = g_1 g_2 g_3 g_4, \ a_3, \cdots, a_{11}.$$

We assign to g_1 the values $2, \cdots, 9$ in turn. Taking first $g_1 = 2$, we enter 6 in the column headed g_9 since $2 \cdot 6 \equiv 1 \pmod{11}$. Then g_2 may take any of the values $2, \cdots, 9$ except 2 and 6. For each value of g_1 except 5 and 9, we may exclude a certain value of g_2, different from g_1 and $1/g_1$, and determined so that $a_4 = a_3 + g_1 g_2 = 0$. The complete list of the 20 sets of distinct a_i, and the values of g_1, g_2, g_3, g_4 leading to them, is as follows:

---

* There are $(p^n - 3)(p^n - 5)(p^n - 7) \cdots 4 \cdot 2$ such sets.

g_1	g_2	g_3	g_4	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
2	3	5	8	3	9	6	4	2	10	5	7	8
2	7	5	4	3	6	10	4	9	2	5	7	8
2	9	4	8	3	10	5	9	2	8	4	6	7
3	5	8	2	4	8	7	5	3	2	6	9	10
3	9	6	7	4	9	6	7	8	5	10	2	3
4	6	5	7	5	7	6	10	3	2	4	8	9
4	8	9	6	5	4	6	7	8	10	9	2	3
5	3	8	2	6	10	9	7	5	4	8	2	3
5	6	4	8	6	3	2	5	8	7	4	9	10
6	3	5	8	7	3	5	10	4	6	2	8	9
6	5	8	4	7	4	2	5	8	6	3	9	10
6	7	5	4	7	5	6	10	3	4	2	8	9
7	3	5	2	8	7	2	3	4	10	9	5	6
7	9	4	6	8	5	4	9	3	2	10	6	7
8	2	5	3	9	3	6	4	2	5	10	7	8
8	9	4	6	9	4	6	7	8	10	5	2	3
9	4	6	7	10	2	9	3	8	4	7	5	6
9	6	3	7	10	9	6	7	8	5	4	2	3
9	6	7	3	10	9	2	3	4	8	7	5	6
9	7	3	2	10	7	9	2	6	8	5	3	4

As a further check, it was verified that each of the resulting 20 sets a_i satisfies conditions (9). Each set leads to a complete set S_{55}^{12}.